Two Lorentzian Lattices

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Abstract

We prove a conjecture formulated in [14], which in turn provides a good deal of evidence for the monstrous proposal of Daniel Allcock [2].

1 Introduction

Let $\mathbb{Z}^{n,1}$ be the odd unimodular Lorentzian lattice with basis $e_0, e_1, \cdots, e_n$ and inner product

$$(e_0, e_0) = -1, (e_0, e_p) = 0, (e_p, e_q) = \delta_{pq}$$

for $p, q = 1, \cdots, n$. Let $\mathbb{R}^{n,1} = \mathbb{R} \otimes \mathbb{Z}^{n,1}$ be the corresponding Lorentzian real vector space, and let

$$\mathbb{R}^{n,1}_+ = \{x \in \mathbb{R}^{n,1}; (x, x) < 0, x_0 > 0\}$$

be the connected component of the light cone complement containing e_0 and let

$$\mathbb{B}^n = \mathbb{R}^{n,1}_+ / \mathbb{R}_+$$

be the real hyperbolic ball of dimension n. The index 2 subgroup of the Lorentz group $\mathrm{O}(\mathbb{R}^{n,1})$ preserving the component $\mathbb{R}^{n,1}_+$ is called the forward Lorentz group and is denoted $\mathrm{O}_+(\mathbb{R}^{n,1})$. Clearly $\mathrm{O}_+(\mathbb{R}^{n,1})$ has two connected components distinguished by the sign of the determinant. It is well known that

$$\Gamma^n = \mathrm{O}_+(\mathbb{Z}^{n,1}) = \mathrm{O}_+(\mathbb{R}^{n,1}) \cap \mathrm{O}(\mathbb{Z}^{n,1})$$

is a discrete subgroup of $\mathrm{O}_+(\mathbb{R}^{n,1})$ acting on $\mathbb{B}^n$ properly discontinuously with cofinite volume. It contains reflections

$$s_\alpha(x) = x - 2(x, \alpha)\alpha/\alpha^2$$

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in roots $\alpha \in \mathbb{Z}^{n,1}$ of norm 1 or norm 2. Here we denote $\alpha^2 = (\alpha, \alpha)$ for the norm of $\alpha \in \mathbb{Z}^{n,1}$. The next theorem is due to Vinberg (for $n \leq 17$ and for $n = 18, 19$ in collaboration with Kaplinskaya) [25].
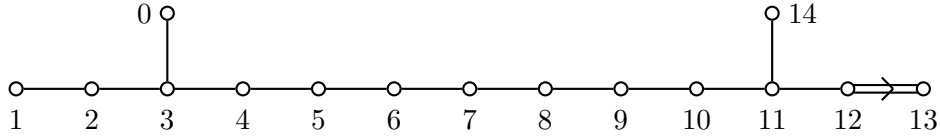
Theorem 1.1. *For $2 \leq n \leq 19$ the group $\Gamma^n = \mathrm{O}_+(\mathbb{Z}^{n,1})$ is generated by refections in roots $\alpha \in \mathbb{Z}^{n,1}$ of norm 1 or norm 2. Moreover the Coxeter diagram for $n = 7$ is given by*
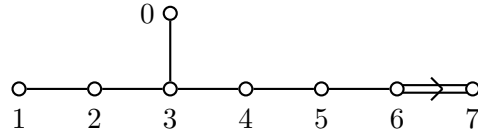
with simple roots

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \cdots, \alpha_6 = e_6 - e_7, \alpha_7 = e_7$$

*and for $n = 13$ the Coxeter diagram takes the form*

with simple roots

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \cdots, \alpha_{11} = e_{11} - e_{12},$$
$$\alpha_{12} = e_{12} - e_{13}, \alpha_{13} = e_{13}, \alpha_{14} = 3e_0 - e_1 - \cdots - e_{11}.$$

Let Γ_1^n be the normal subgroup of Γ^n generated by the reflections in norm 1 roots. Clearly Γ_1^n is a subgroup of the principal congruence subgroup $\Gamma^n(2)$ of level 2, and in fact we have equality if and only if $n \leq 7$ as shown by Everitt, Ratcliffe and Tschantz [13] (see [16] for a quick proof). Let D be the closed fundamental chamber in the closure of $\mathbb{R}_+^{n,1}$ for Γ^n containing the point $e_0 - (\epsilon_1 e_1 + \cdots + \epsilon_n e_n)$ in $\mathbb{R}_+^{n,1}$ (for $\epsilon_1 > \cdots > \epsilon_n > 0$ all small), in accordance with the choice of positive roots in case $n = 7, 13$ in the above theorem, and let $G \supset D$ be the closed fundamental chamber for the Coxeter group Γ_1^n. By Coxeter group theory

$$G = \cup_{\gamma \in \Gamma_0^n} \gamma D$$

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with Γ_0^n the subgroup of Γ^n generated by the simple roots of norm 2, and $\Gamma^n = \Gamma_1^n \rtimes \Gamma_0^n$. The local structure of x in G near $x_0 e_0$ with $x_0 > 0$ is given by $x_p \leq 0$ for $p = 1, \cdots, n$.

The Fano plano is the projective plane $\mathbb{P}^2(2)$ over a field of 2 elements. It has 7 points, denoted $\{a, a_i, c_i\}$ with $i = 1, 2, 3$, and 7 lines, given by the line z through $\{c_1, c_2, c_3\}$, the three lines $\{b_i\}$ through $\{a, a_i, c_i\}$ and the remaining three lines $\{d_i\}$ through $\{c_i, a_j, a_k\}$ with $\{i, j, k\} = \{1, 2, 3\}$ in accordance with the picture below on the left. The incidence graph I_{14} is bipartite with 7 black nodes from the set of points $\mathcal{P}$ and 7 white nodes from the set of lines $\mathcal{L}$. In its picture below on the right an ordinary bond means incidence, while the thick bond between a_i and d_i indicates that a_i and d_j are incident (and so connected) if and only if $i \neq j$ [23].
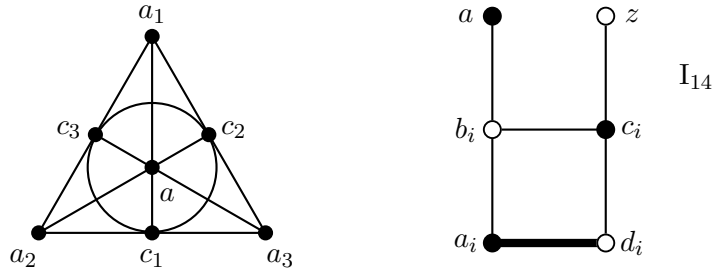
The group of diagram automorphism $\mathrm{PGL}_3(2) \cdot 2$ of the Coxeter diagram I_{14} has the simple index 2 subgroup $\mathrm{PGL}_3(2) \cong \mathrm{PSL}_2(7)$ of order 168 as colour preserving automorphisms. The remaining group elements are involutions corresponding to projective dualities and interchange black and white nodes.

Theorem 1.2. *Fix a bijection $\mathcal{P} \to \{1, \cdots, 7\}$ and so $e_p \in \mathbb{Z}^{7,1}$ for $p \in \mathcal{P}$. Let us write $e_l = e_0 - \sum_{p \in l} e_p \in \mathbb{Z}^{7,1}$ for $l \in \mathcal{L}$, and therefore $(e_p, e_q) = \delta_{pq}$ for all $p, q \in \mathcal{P}$, $(e_l, e_m) = 2\delta_{lm}$ for all $l, m \in \mathcal{L}$, and $(e_p, e_l) = -1, 0$ if $p \in l, p \notin l$ respectively. If we denote*

$$P = \{x \in \mathbb{R}^{7,1}; (x, e_p) \leq 0, (x, e_l) \leq 0 \text{ for all } p \in \mathcal{P}, l \in \mathcal{L}\}$$

then the induced hyperbolic polytope in $\mathbb{B}^7$ has finite volume and we have the inclusions $D \subset P \subset G$.

The proof of this result is a rather straightforward calculation given in the next section. The interest of this theorem is its analogy with the theorem below, which was conjectured in [14] and provides a positive step towards the monstrous proposal of Daniel Allcock [2].

Now, let us consider the projective plane $\mathbb{P}^2(3)$ over a field of 3 elements, and denote as before by $\mathcal{P}$ and $\mathcal{L}$ the sets of its 13 points and 13 lines respectively. Its incidence graph I_{26} with 26 nodes indexed by $\mathcal{P} \sqcup \mathcal{L}$ has a picture given below with the same convention on the meaning of the thick bonds and the index $i \in \{1, 2, 3\}$ as before [11]. Its automorphism group is the group $\mathrm{PGL}_3(3) \cdot 2$ of order $11232 = 2^5 \cdot 3^3 \cdot 13$ consisting of projective transformations and projective dualities.
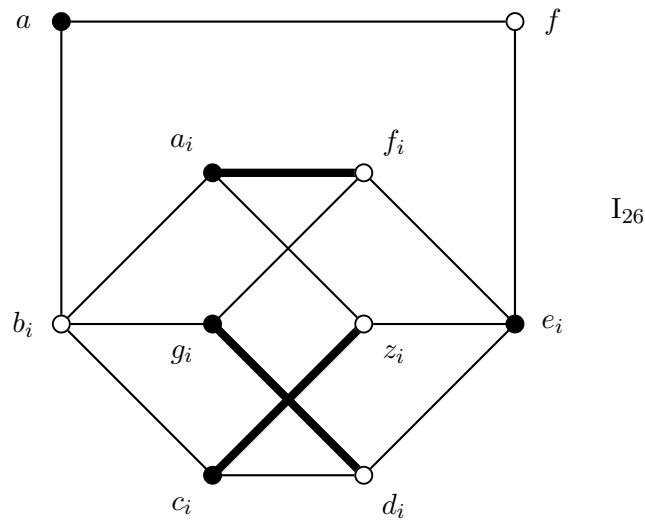
The next result is the exact analogue of the previous theorem for the projective plane of order 3 rather than 2.

Theorem 1.3. *Fix a bijection $\mathcal{P} \to \{1, \cdots, 13\}$ and write $e_l = e_0 - \sum_{p \in l} e_p$ for $l \in \mathcal{L}$, and therefore $(e_p, e_q) = \delta_{pq}$ for all $p, q \in \mathcal{P}$, $(e_l, e_m) = 3\delta_{lm}$ for all $l, m \in \mathcal{L}$, and $(e_p, e_l) = -1, 0$ if $p \in l, p \notin l$ respectively. If we denote*

$$P = \{x \in \mathbb{R}^{13,1}; (x, e_p) \leq 0, (x, e_l) \leq 0 \text{ for all } p \in \mathcal{P}, l \in \mathcal{L}\}$$

then the induced hyperbolic polytope in $\mathbb{B}^{13}$ has finite volume and we have the inclusion $P \subset G$.

Again the proof is by straightforward but more extensive (however not unpleasant) calculations. The inclusion $P \subset G$ was conjectured in [14] and in the final section of this paper we discuss its relevance towards the monstrous proposal of Daniel Allcock [2].

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2 Proof of Theorem 1.2

In this section we consider the case $n = 7$ and I_{14} is the incidence graph of $\mathbb{P}^2(2)$ with black nodes from the set of points $\mathcal{P}$ and white nodes from the set of lines $\mathcal{L}$. The extremals of the fundamental chamber D for the Coxeter group Γ^7 are spanned over $\mathbb{R}_+$ by

$$v_0 = e_0, v_1 = e_0 - e_1, v_2 = 2e_0 - e_1 - e_2, v_p = 3e_0 - e_1 - \cdots - e_p$$

for $p = 3, \cdots, 7$ as antidual basis of the basis of simple roots in Theorem 1.1.

The group Γ_0^7 is the Weyl group $W(E_7)$ as stabilizer of the vector v_7. Note that the fundamental chamber G for the Coxeter group Γ_1^7 is bounded by 56 (being $|W(E_7)|/|W(E_6)|$) walls corresponding to the positive norm 1 roots

$$e_p, e_0 - e_p - e_q, 2e_0 - e_1 - \cdots - e_7 + e_p + e_q, 3e_0 - e_1 - \cdots - e_7 - e_p$$

for $p, q \in \mathcal{P}$ with $p \neq q$, making all together $7 + \binom{7}{2} + \binom{7}{2} + 7 = 56$ roots as should. This description is well known from the description of the 56 lines on a degree two del Pezzo surface [12]. All these simple roots have the same inner product with the vector v_7, which for that reason is called a Weyl vector for the chamber G.

The convex cone

$$P = \{x \in \mathbb{R}^{7,1}; (x, e_p) \leq 0, (x, e_l) \leq 0 \text{ for all } p \in \mathcal{P}, l \in \mathcal{L}\}$$

has dihedral angles $\pi/4$ or $\pi/2$ and the reflections in the walls of P generate a Coxeter group with Coxeter diagram I_{14}, but with all edges marked with 4. Its connected parabolic subdiagrams are of type A_3, and each such is contained in a parabolic subdiagram of type $3A_3$ (in our picture of the Coxeter diagram I_{14} we left out the marks 4 on the bonds, and here we continue this convention, so A_3 actually stands for $\tilde{C}_2$ and this is indeed parabolic). Since the Coxeter diagram has no Lannér subdiagrams we conclude that P has finite hyperbolic volume by the Vinberg criterion [15]. Since the inclusion $D \subset P$ is trivial by looking around $v_\mathcal{P}$ it remains to show that $P \subset G$. Denote

$$v_\mathcal{P} = v_0, v_\mathcal{L} = v_7/\sqrt{2}$$

for two norm -1 vectors in $\mathbb{R}_+^{7,1}$, given as the intersection of the mirrors perpendicular to the norm 1 roots e_p for $p \in \mathcal{P}$ and the intersection of the mirrors for the norm 2 roots $e_l = e_0 - \sum_{p \in l} e_p$ for $l \in \mathcal{L}$ respectively.

Lemma 2.1. *The actual vertices of the hyperbolic polytope obtained from P are represented by the 2 vectors $v_{\mathcal{P}}, v_{\mathcal{L}}$ together with the 56 vectors*

$$v_{p,l} = 2e_0 - \sum_{q \notin l \sqcup \{p\},} e_q, v_{l,p} = (3e_0 - 2e_p - \sum_{q \in l} e_q)/\sqrt{2}$$

for all $p \in \mathcal{P}, l \in \mathcal{L}$ with $p \notin l$. The ideal vertices are represented by the 14 vectors

$$u_p = e_0 - e_p, u_l = (2e_0 - \sum_{p \notin l} e_p)/\sqrt{2}$$

for all $p \in \mathcal{P}, l \in \mathcal{L}$.

Proof. Since the hyperbolic polytope obtained from P is a finite volume convex acute angled polytope its actual vertices are given by the elliptic subdiagrams of its Coxeter diagram of maximal rank 7. A connected elliptic subdiagram of the diagram I_{14} with all edges marked with the number 4 (however, in all figures above and below this mark 4 will be deleted) is of type A_1 or A_2. For example, if you leave out from I_{14} the black node a then the maximal subdiagram obtained by also deleting the white nodes b_i connected with a has the left form
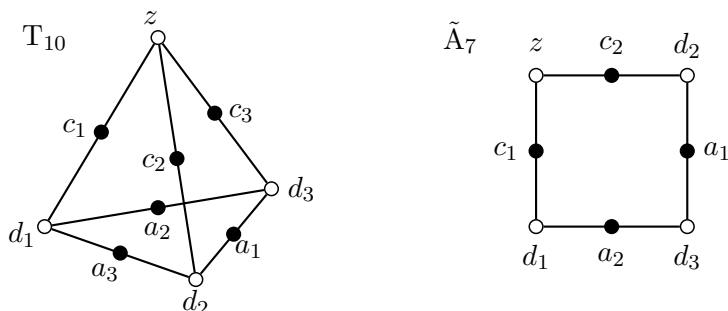
and so is a tetrahedron with white nodes at the 4 vertices and black nodes at the midpoints of 6 egdes. We denote this Coxeter diagram by T_{10} and by straightforward inspection it has only one elliptic subdiagram of rank 6, namely of type $6A_1$ and consisting of the 6 black nodes. Hence we recover the subdiagram of type $7A_1$ in I_{14} consisting of the 7 black nodes corresponding to the vertex $v_{\mathcal{P}}$ of P. Similarly the subdiagram of type $7A_1$ in I_{14} consisting of the 7 white nodes corresponds to the vertex $v_{\mathcal{L}}$.

The other possibility is that we start with a subdiagram of type A_2 of I_{14}, for example the subdiagram with nodes $\{a, b_3\}$. Leaving out the

connecting nodes $\{b_1, b_2, a_3, c_3\}$ gives us a diagram of type $\tilde{A}_7$ (also called a free octagon) and drawn on the right. Any elliptic subdiagram of rank 5 of this free octagon is of type $A_1 \sqcup 2A_2$, and so we get an elliptic subdiagram of I_{14} of type $A_1 \sqcup 3A_2$. These give the 28 vertices $v_{p,l}$ as given above with $p \in \mathcal{P}, l \in \mathcal{L}, p \notin l$. A projective duality interchanges $e_0 = v_{\mathcal{P}}$ with $v_{\mathcal{L}} = v_7/\sqrt{2}$, and the e_p with the vectors $e_l/\sqrt{2} = (e_0 - \sum_{q \in l} e_q)/\sqrt{2}$ for $p \in \mathcal{P}, l \in \mathcal{L}$. By direct calculation the vertices $v_{p,l}$ give by projective duality the 28 vertices $v_{l,p}$ as given above.

The ideal vertices of P correspond to the maximal parabolic subdiagrams of type $3A_3$ of rank 6, and the 14 vertices u_p and u_l for $p \in \mathcal{P}, l \in \mathcal{L}$ follow by a direct computation. $\qquad\square$

Remark 2.2. *The vector $v_{\mathcal{P}} = v_0 = e_0$ is a vertex of all three hyperbolic polytopes associated with D, P, G and the local structure near this vertex of P and G coincides and is equal to $\cup_\gamma \gamma D$ by letting γ run over the symmetric group $S_7 = W(A_6)$ as stabilizer of the edge of D from v_0 to v_7. There are 7 edges of P and G from v_0 to the 7 ideal vertices $u_p = e_0 - e_p$. There are $\binom{7}{2}$ faces of dimension 2 of P and G containing v_0, which are right angled triangles at v_0 with 2 ideal vertices u_p, u_q for $p, q \in \mathcal{P}$ distinct. There are $\binom{7}{3}$ faces of dimension 3 of G containing v_0, which are double tetrahedra with 3 ideal vertices u_p, u_q, u_r for $p, q, r \in \mathcal{P}$ distinct and just one more actual vertex $2e_0 - e_p - e_q - e_r$ obtained from v_0 by reflection in the norm 2 root $e_0 - e_p - e_q - e_r$ [16]. Apparently only those for which p, q, r are not collinear give $7 \cdot 6 \cdot 4/3! = 28$ faces of dimension 3 for P leading to the vertices $v_{p,l}$ as in the lemma. The remaining vertices of P are obtained by projective duality.*

The proof of the inclusion $P \subset G$ as stated in Theorem 1.2 follows now by direct inspection. The vertices $v_{\mathcal{P}}, v_{p,l}$ and u_p for $p \in \mathcal{P}, l \in \mathcal{L}, p \notin l$ are also vertices of G. By direct inspection the four types of simple roots

$$e_p, e_0 - e_p - e_q, 2e_0 - e_1 - \cdots - e_7 + e_p + e_q, 3e_0 - e_1 - \cdots - e_7 - e_p$$

for the Gosset chamber G take values from

$$\{-2, -1, 0\}, \{-3, -2, -1, 0\}, \{-4, -3, -2, -1\}, \{-4, -3, -2\}$$

on the actual vertices $\sqrt{2} v_{l,p} = 3e_0 - 2e_p - \sum_{q \in l} e_q$ of P respectively, and values from

$$\{-1, 0\}, \{-2, -1, 0\}, \{-2, -1, 0\}, \{-2, -1\}$$

on the ideal vertices $\sqrt{2} u_l$ of P respectively. Finally these simple roots take the single value -1 on the Weyl vector $v_{\mathcal{L}}$ of G. Since all these numbers are

≤ 0 we conclude that all the vertices of P are contained in G, which by the Minkowski convex hull theorem implies $P \subset G$.

The hyperbolic polytope G has $576 = 2^6 \cdot 3^2$ (being $|W(\mathrm{E}_7)|/|W(\mathrm{A}_6)|$) actual vertices, which equals $2^7(1 + 28/8)$ as should. Indeed, the hyperbolic polytope G has a tiling by 2^7 congruent copies of P meeting all at $v_{\mathcal{L}}$, while just of them one contains $v_{\mathcal{P}}$, and 2^3 of them meet at each $v_{p,l}$.

3 An odd presentation for $W(\mathrm{E}_7)$

The Weyl group $W(\mathrm{E}_6)$ has a remarkable presentation due to Christopher Simons [24] as factor group of the hyperbolic Coxeter group $W(\mathrm{P}_{10})$ modulo deflation of the free hexagons. Here P_{10} is our notation for the Petersen graph. This presentation of $W(\mathrm{E}_6)$ was given a geometric explanation [16] using the Allcock–Carlson–Toledo period map for the moduli space of cubic surfaces [4] and its reality analysis by Yoshida [26]. In fact, Yoshida only discussed the maximal real component of real cubic surfaces with 27 real lines, which is also the component we used. The complete reality analysis for the 5 components with $27, 15, 7, 3, 3$ real lines respectively was worked out in [5].
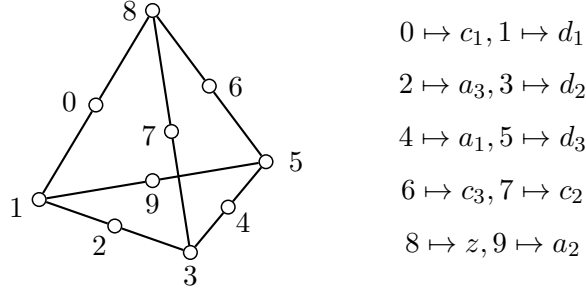
In this section we tell a similar story for $W(\mathrm{E}_7)$ using the Kondo period map for the moduli space of quartic curves [19] and its reality analysis by Heckman and Rieken [21],[17]. Again only the maximal real component of real quartic curves with 4 components or equivalently with 28 real bitangents is needed for this purpose.

Theorem 3.1. *If T_{10} is the tetrahedral Coxeter diagram (but this time bonds do have mark 3 rather than 4, and so are deleted as usual), then the Weyl group $W(\mathrm{E}_7)$ has a presentation as factor group of the hyperbolic Coxeter group $W(\mathrm{T}_{10})$ modulo deflation of the free octagons.*

Proof. Consider the Coxeter diagram
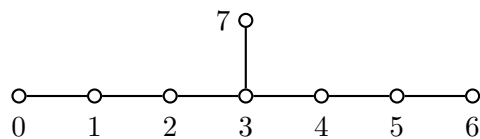
of type $\tilde{\mathrm{E}}_7$ as subdiagram of T_{10} via the numeration

$$0 \mapsto c_1, 1 \mapsto d_1$$
$$2 \mapsto a_3, 3 \mapsto d_2$$
$$4 \mapsto a_1, 5 \mapsto d_3$$
$$6 \mapsto c_3, 7 \mapsto c_2$$
$$8 \mapsto z, 9 \mapsto a_2$$

Let $\alpha_1, \cdots, \alpha_7$ be the simple roots in $R_+(\mathrm{E}_7)$ and put

$$\begin{aligned}
\alpha_0 &= -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7) \\
\alpha_8 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_7 \\
\alpha_9 &= \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7
\end{aligned}$$

as vectors in $R(\mathrm{E}_7)$. Then we have $(\alpha_x, \alpha_y) = 0$ for any two disconnected nodes x, y of T_{10}, while $(\alpha_x, \alpha_y) = -1$ for any two connected nodes x, y of T_{10}, except for $\{x, y\} = \{7, 8\}, \{5, 9\}$ in which case this value is 1. Hence the group $W(\mathrm{T}_{10})$ modulo deflation of the free octagons admits an epimorphsm onto $W(\mathrm{E}_7)$.

There are three free hexagons, obtained by leaving out the midpoints $\{0, 4\}, \{2, 6\}, \{7, 9\}$ of pairs of opposite edges, and their deflations amount to the relations

$$s_1 s_2 s_3 s_7 s_8 s_6 s_5 s_9 = 1, s_1 s_9 s_5 s_4 s_3 s_7 s_8 s_0 = 1, s_1 s_2 s_3 s_4 s_5 s_6 s_8 s_0 = 1$$

respectively. We claim that the generators s_8, s_9, s_0 are superfluous. Indeed s_9 is a word in $s_1, \cdots, s_7$ using the second and the third relation. By the first relation this implies that s_8 is also a word in $s_1, \cdots, s_7$. Finally by the second relation we conclude that s_0 is also a word in $s_1, \cdots, s_7$. This proves that the Coxeter group $W(\mathrm{T}_{10})$ modulo deflation of the free octagons is indeed isomorphic to $W(\mathrm{E}_7)$. $\qquad \square$

The Coxeter diagram T_{10} has the group S_4 as group of diagram automorphisms, and the deflation relations are also preserved. In turns out that this S_4 is a subgroup of $W(\mathrm{E}_7)$ acting by inner automorphism on the generators. The root system $R(\mathrm{E}_7)$ has a subroot system of type $6R(\mathrm{A}_1)$, which is unique up to conjugation by $W(\mathrm{E}_7)$ and is normalized by the simple group $\mathrm{PGL}_3(2)$ of order 168. This realizes the semidirect product $2^6 \rtimes \mathrm{PGL}_3(2)$ as a maximal subgroup of $W(\mathrm{E}_7)$, and this monomorphism is unique up to conjugation. The group S_4 is a maximal subgroup of $\mathrm{PGL}_3(2)$, unique

up to (inner and outer) automorphisms of $\mathrm{PGL}_3(2)$. In turn this realizes S_4 as subgroup of $W(\mathrm{E}_7)$ in a up to conjugation almost unique way, the difference between the 2 monomorphisms is coming from the center ± 1 of $W(\mathrm{E}_7)$. This monomorphism $S_4 \hookrightarrow W(\mathrm{E}_7)$ acts on the odd presentation of $W(\mathrm{E}_7)$ as group of diagram automorphisms.

Consider the extended tetrahedral Coxeter diagram T_{13} with 4 white nodes corresponding to norm 2 vectors and 9 black nodes corresponding to norm 1 vectors in $\mathbb{Z}^{6,1}$. The ordinary branches mean that the inner product is $-\sqrt{2}$ so the the dihedral angle between the 2 mirrors is $\pi/4$, while the dashes branches mean that the inner product is -1 so the 2 mirrors are parallel. But for simplicity we leave out the marks 4 and ∞ from the branches as before. It is obtained from the tetrahedral diagram T_{10} in the proof of Lemma 2.1 by adding 3 black nodes $\{b_i'\}$ dashed connected to the midpoints $\{a_i, c_i\}$ of pairs of opposite edges of the tetrahedron, while the 3 nodes $\{b_i'\}$ are also dashed connected among each other. From this description it is clear that the Coxeter diagram T_{13} has S_4 as group of diagram automorphisms.

This diagram has parabolic subdiagrams of type $\tilde{\mathrm{A}}_1 \sqcup 2\tilde{\mathrm{C}}_2$ of rank 5 and no Lannér subdiagrams. Hence the corresponding Coxeter group $W(\mathrm{T}_{13})$ has cofinite volume on hyperbolic space $\mathbb{B}^6$ by the Vinberg criterion. Its fundamental chamber P' is just the wall of the fundamental chamber P corresponding to the incidence Coxeter diagram I_{14} of the Fano plane as described in the introduction. Indeed the roots with index b_i' are obtained from those by b_i in the Coxeter diagram I_{14} by orthogonal projection on the orthogonal complement of the root with index a.
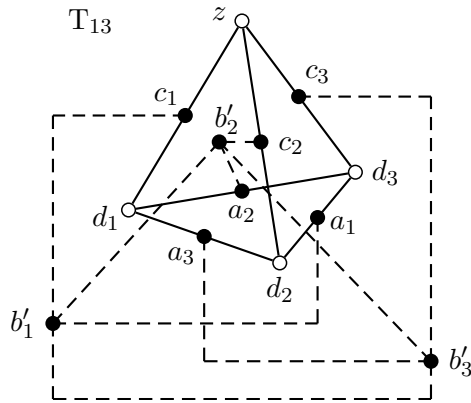
The hyperbolic polytope in $\mathbb{B}^6$ associated with P' has symmetry group S_4 and the interior points of its quotient by S_4 corresponds to the moduli space

of maximal real quartic curves, that is real smooth quartic curves with 4 connected components. For an analysis of the real algebraic geometry of the Kondo period map [19] we refer to Section 5.7 of the PhD thesis by one of us [21]. The group S_4 acts as symmetry group of the 4 connected components of the maximal real quartic curve, and hence we find an up to conjugation unique monomorphism $S_4 \hookrightarrow W(\mathrm{E}_7)/\{\pm 1\}$ in the symmetry group of the 28 bitangent of the quartic curve. The conclusion is that the Weyl groups $W(\mathrm{E}_6)$ and $W(\mathrm{E}_7)$ have analogous odd presentations as factor groups of the hyperbolic Coxeter groups $W(\mathrm{P}_{10})$ and $W(\mathrm{T}_{10})$ modulo deflation of the free hexagons and octagons respectively, and these presentations have natural geometric meaning coming from the action of these Weyl groups on the moduli spaces of marked maximally real del Pezzo surfaces of degree 3 and 2 respectively.

4   Proof of Theorem 1.3

In this section let $n = 13$ and let I_{26} be the incidence graph of the projective plane $\mathbb{P}^2(3)$ over a field of 3 elements. Clearly the chamber G is invariant under the symmetric group S_{13} generated by the simple roots $s_1, \cdots, s_{12}$ in the notation of Theorem 1.1. The connected elliptic subdiagrams (relative to the hyperbolic metric) of I_{26} are of type A_k for $k = 1, 2, 3, 4$, while the connected parabolic subdiagrams are of type A_5 and D_4 of rank 4 and 3 respectively. In order to arrive at the analogue of Lemma 2.1 we have the following combinatorial lemma.

Lemma 4.1. *The elliptic subdiagrams of* I_{26} *of maximal rank* 13 *are of type* $13\mathrm{A}_1$ *or of type*

$$4\mathrm{A}_1 \sqcup 3\mathrm{A}_3, \mathrm{A}_1 \sqcup 4\mathrm{A}_3, 2\mathrm{A}_1 \sqcup \mathrm{A}_2 \sqcup 3\mathrm{A}_3, \mathrm{A}_1 \sqcup 3\mathrm{A}_4, \mathrm{A}_2 \sqcup \mathrm{A}_3 \sqcup 2\mathrm{A}_4, 3\mathrm{A}_3 \sqcup \mathrm{A}_4$$

while the parabolic subdiagrams of maximal rank 12 *are of type* $3\mathrm{A}_5$ *or* $4\mathrm{D}_4$. *Any two elliptic subdiagrams of* I_{26} *of maximal rank* 13 *and of the same type are conjugated under the automorphism group* $\mathrm{PGL}_3(3) \cdot 2$.

Proof. It is easy to check that all elliptic subdiagrams of rank 13 consisting of only type A_1 factors are those consisting of all black or all white nodes. Indeed, if there is 1 white (line) node, then there are at most $9 = 13 - 4$ black (point) nodes, and so together at most 10 nodes. If there are 2 white nodes, then there are at most $6 = 13 - 7$ black nodes, and so together at most 8 nodes. If there are 3 white nodes, then there are at most $4 = 13 - 9$ black nodes, and so together at most 7 nodes. If there are 4 white nodes, then

there are at most 3 black nodes, and so together at most 7 nodes. Hence in all these cases we never arrive at 13 nodes, and so the only cases left of type $13A_1$ are all black nodes or all white nodes.

Suppose on the other extreme that we have an elliptic subdiagram of maximal rank 13 with at least one component of type A_4 (any two of these are conjugate under the group $PGL_3(3)$ of colour preserving automorphisms of I_{26}). The Coxeter diagram obtained by deleting this A_4 together with all nodes connected to this A_4 and all bonds connected to these nodes is a free dodecagon of type $\tilde{A}_{11}$. Hence we have to look for an elliptic subdiagram of $\tilde{A}_{11}$ of maximal rank 9. This amounts to finding partitions $n_1 + n_2 + n_3$ of 12 into three parts $n_i \leq 5$, which are just the three possibilities

$$\{n_i\} = \{2, 5, 5\}, \{3, 4, 5\}, \{4, 4, 4\}$$

correspondinig to the above types. This leads to the three types of elliptic subdiagrams of I_{26} of maximal rank 13 with at least one component of type A_4 as given in the lemma.

Now suppose we have an elliptic subdiagram of I_{26} of maximal rank 13 with at least one factor of type A_3 and the other components of type A_k with $k \leq 3$. The maximal Coxeter diagram of I_{26} disconnected from this A_3 is a free dodecagon (with nodes numbered $1, \cdots, 12$) together with three more nodes, each node attached to an opposite pair of odd numbered nodes of the free dodecagon. Let us denote this diagram by I_{15}.

The I_{15} diagram has two essentially different subdiagrams of type A_3, whose maximal disconnected complement is either a free octagon of type $\tilde{A}_7$ or is of type $\tilde{D}_8$. The diagram $\tilde{A}_7$ has no elliptic subdiagrams of rank 7, but the diagram $\tilde{D}_8$ has three elliptic subdiagrams of rank 7 with only components of type A_k for $k \leq 3$, namely of type $4A_1 \sqcup A_3$, $A_1 \sqcup 2A_3$ and $2A_1 \sqcup A_2 \sqcup A_3$.

The diagram of type I_{15} has essentially one subdiagram of type A_2, whose maximal disconnected complement is just a free octagon of type $\tilde{A}_7$ together with two more nodes, one attached to a node of the free octagon and the other attached to the opposite node of the free octagon. This diagram has just one elliptic subdiagram of rank 8 with only components of type A_k for $k \leq 3$, namely the one of type $2A_1 \sqcup 2A_3$.

Since I_{15} has no elliptic subdiagram of type $10A_1$ we find that the I_{26} diagram has just the three types of elliptic subdiagrams of rank 13 with at least one component of type A_3 and the others of type A_k with $k \leq 3$ as given in the lemma.

Now suppose we have an elliptic subdiagram of maximal rank 13 with at least one factor of type A_2. Any two subdiagrams of type A_2 are conjugate

under the group $\mathrm{PGL}_3(3)$, so we may consider the one with nodes $\{a, f\}$. The maximal disconnected subdiagram in I_{26} has the shape
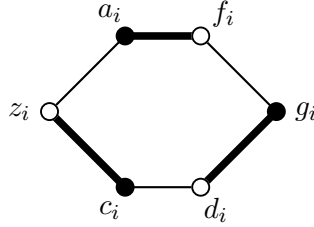
which we denote by I_{18}. This is a trivalent graph with 18 nodes and we need to look for its elliptic subdiagrams of rank 11. Such an elliptic subdiagram is not possible if all connected components of this elliptic subdiagram are of type A_k with $k \leq 2$. For example, an elliptic subdiagram of type $A_1 \sqcup 5A_2$ is impossible, since we need at least $(3 + 5.4)/3 > 7$ additional nodes, which all together makes more than the total available 18 nodes. Hence the I_{26} diagram has no elliptic subdiagram of rank 13 with one component of type A_2 and all others of type A_k with $k \leq 2$.

The case of parabolic subdiagrams of I_{26} of maximal rank 12 is easy, and gives the two possibilities as stated in the lemma. $\qquad\square$

The next step is an explicit description of the actual and ideal vertices of the finite volume hyperbolic polytope obtained from the convex cone

$$P = \{x \in \mathbb{R}^{13,1}; (x, e_p) \leq 0, (x, e_l) \leq 0 \text{ forall } p \in \mathcal{P}, l \in \mathcal{L}\}$$

corresponding to the various types described in the previous lemma. As before we denote by $e_l = e_0 - \sum_{p \in l} e_p$ for $l \in \mathcal{L}$ the simple norm 3 roots. A projective duality in $\mathrm{PGL}_3(3) \cdot 2$ corresponds to an isometry

$$v_{\mathcal{P}} = e_0 \mapsto v_{\mathcal{L}} = (4e_0 - \sum_{p \in \mathcal{P}} e_p)/\sqrt{3}, e_p \mapsto e_l/\sqrt{3}$$

as before. The analogue of Lemma 2.1 gives the following description.

Lemma 4.2. *The actual vertices of the hyperbolic polytope obtained from P are represented by the vectors*

$$v_{\mathcal{P}} = e_0, v_{\mathcal{L}} = (4e_0 - \sum_{q \in \mathcal{P}} e_q)/\sqrt{3}$$

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of type $13A_1$*, and by the vectors*

$$v_{pqr} = 2e_0 - (e_p + e_q + e_r), v_{lmn} = (5e_0 - 2\sum_{s\notin l\cup m\cup n} e_s - \sum_{s\in(l\cup m\cup n)'} e_s)/\sqrt{3}$$

of type $4A_1 \sqcup 3A_3$ *for* $p, q, r \in \mathcal{P}$ *not all on a line and* $l, m, n \in \mathcal{L}$ *not all through a point with* $(l \cup m \cup n)'$ *the set of 6 points on exactly one of these 3 lines, and by the vectors*

$$v_{p,l} = (3e_0 - \sum_{q\notin l\sqcup p} e_q), v_{l,p} = (4e_0 - 3e_p - \sum_{q\in l} e_q)/\sqrt{3}$$

of type $A_1 \sqcup 4A_3$ *for all* $p \in \mathcal{P}, l \in \mathcal{L}$ *with* $p \notin l$*, and by the vectors*

$$v_{p,q,l} = 3e_0 - 2e_p - \sum_{r\in l, r\neq q} e_r, v_{l,m,p} = (7e_0 - 2\sum_{r\notin l\sqcup p} e_r - \sum_{r\in m, r\neq p} e_r)/\sqrt{3}$$

of type $2A_1 \sqcup A_2 \sqcup 3A_3$ *for all* $p, q \in \mathcal{P}$ *and* $l, m \in \mathcal{L}$ *with* $p \notin l, q \in l$ *and likewise* $p \notin l, p \in m$*, and by the vectors*

$$v_{p,qrs} = 4e_0 - 2(e_q + e_r + e_s) - (e_u + e_v + e_w)$$
$$v_{k,lmn} = (7e_0 - 4e_p - 3(e_q + e_r + e_s) - (e_x + e_y + e_z))/\sqrt{3}$$

of type $A_1 \sqcup 3A_4$ *for all* $p, q, r, s \in \mathcal{P}$ *in general position (no three on a line) with* $u \in l_{pq} - l_{rs} - \{p, q\}$*,* $v \in l_{pr} - l_{qs} - \{p, r\}$*,* $w \in l_{ps} - l_{qr} - \{p, s\}$ *for* $v_{p,qrs}$ *and likewise for all* $k, l, m, n \in \mathcal{L}$ *in general position (no three through a point) with* k *the line through* p *but not through* q, r, s *and* $x = k \cap l_{rs}$*,* $y = k \cap l_{qs}$*,* $z = k \cap l_{qr}$ *for* $v_{k,lmn}$ *(see picture below), and by the vectors*

$$v_{p,qr,s} = 5e_0 - 3e_p - 2(e_q + e_r + e_s) - (e_x + e_y)$$
$$v_{k,lm,n} = (9e_0 - 5e_p - 4(e_r + e_s) - 3n_q - 2(n_y + n_z) - n_h)/\sqrt{3}$$

of type $A_2 \sqcup A_3 \sqcup 2A_4$ *for all* $p, q, r, s \in \mathcal{P}$ *in general position for* $v_{p,qr,s}$ *and likewise for all* $k, l, m, n \in \mathcal{L}$ *in general position with* $h = l_{pq}\cap l_{rs}$*,* $i = l_{pr}\cap l_{qs}$*,* $j = l_{ps} \cap l_{qr}$ *for* $v_{k,lm,n}$ *(see picture below), and by the vectors*

$$v_{p,q,rs} = 3e_0 - 2e_p - (e_q + e_r + e_s + e_x)$$
$$v_{k,l,mn} = (6e_0 - 3(e_q + e_r) - 2(e_s + e_u + e_v) - (e_w + e_h + e_i))/\sqrt{3}$$

of type $3A_3\sqcup A_4$ *for all* $p, q, r, s \in \mathcal{P}$ *in general position for* $v_{p,q,rs}$ *and likewise for all* $k, l, m, n \in \mathcal{L}$ *in general position for* $v_{k,l,mn}$ *(see picture below).*
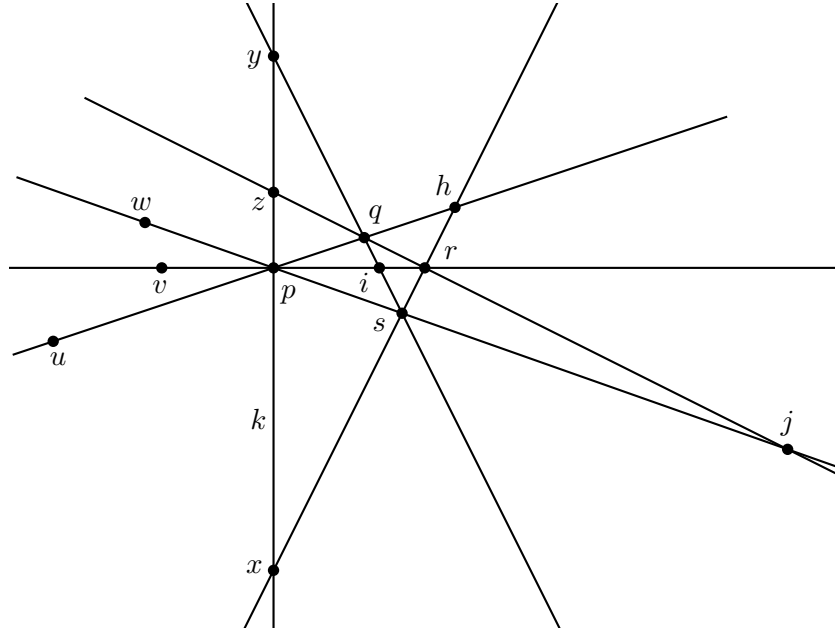
The ideal vertices are represented by the vectors

$$u_p = e_0 - e_p, u_l = (3e_0 - \sum_{p \notin l} e_p)/\sqrt{3}$$

of type $4D_4$ *for all* $p \in \mathcal{P}, l \in \mathcal{L}$, *and by the vectors*

$$u_{pqrs} = 2e_0 - (e_p + e_q + e_r + e_s), u_{klmn} = (4e_0 - 2\sum_p e_p - \sum_q e_q)/\sqrt{3}$$

of type $3A_5$ *for any* $p, q, r, s \in \mathcal{P}$ *in general position and any* $k, l, m, n \in \mathcal{L}$ *in general position with* $\sum_p$ *is the sum over the three* $p \in \mathcal{P}$ *on none of the lines* k, l, m, n *and* $\sum_q$ *is the sum over the four* $q \in \mathcal{P}$ *on exactly one of the lines* k, l, m, n.

Proof. The vertices $v_{\mathcal{P}}, v_{p,l}, u_p$ are analogous to the vertices described in Lemma 2.1, and likewise $v_{\mathcal{L}}, v_{l,p}, u_l$ are found by projective duality. It is obvious that $v_{\mathcal{P}}$ is of type $13A_1$ as all black nodes in I_{26}. It is also clear from the picture below that u_p is of type $4D_4$ (namely $\{l_i, q_i, r_i, s_i\}$ for $i = 1, 2, 3, 4$) and that $v_{l,p}$ is of type $A_1 \sqcup 4A_3$ (namely $\{l\} \sqcup \{l_i, q_i, r_i\}$ for $i = 1, 2, 3, 4$).
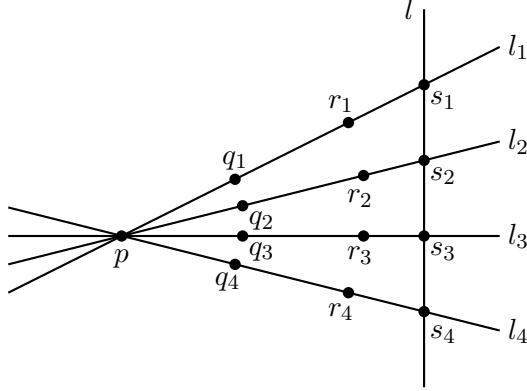
We shall describe the details for finding the ideal vertices u_{pqrs} of type $3A_5$. Consider the maximal tree subdiagram of I_{26} of type Y_{555} with nodes $\{f, e_i, d_i, c_i, b_i, a_i\}$ for $i = 1, 2, 3$. There are 4 remaining black nodes in the I_{26} diagram, namely $p = a, q = g_1, r = g_2, s = g_3$. It is now obvious from the I_{26} diagram that $(u_{pqrs}, e_x) = 0$ for all 9 black (point) nodes x of the Y_{555} subdiagram, and also

$$(u_{pqrs}, e_y) = (u_{pqrs}, e_0 - \sum_{x \in y} e_x) = -2 + |\{p, q, r, s\} \cap y| = 0$$

for all 6 white (line) nodes $y \neq f$ of the Y_{555} subdiagram. The other ideal vertices u_{klmn} are obtained by projective duality.

Similar straightforward calculations work for the remaining actual vertices, for example using that the elliptic subdiagrams

$$A_1 \sqcup 3A_4, A_2 \sqcup A_3 \sqcup 2A_4, 3A_3 \sqcup A_4$$

of I_{26} are all realized within a Y_{555} subdiagram. $\square$

In order to show that the chamber P for the diagram I_{26} is contained in the chamber G for the norm one roots we check that all the vertices of P are contained in G. Since G is invariant under the symmetric group $S_{13} = W(A_{12})$ (with A_{12} the subdiagram of the second Coxeter–Vinberg diagram in Theorem 1.1 with nodes numbered $1, \cdots, 12$) this amounts to checking that all conjugates under S_{13} of the 12 actual and 4 ideal vertices enumerated in the previous lemma lie in G.

Lemma 4.3. *All vertices of P given in the previous lemma are contained in the fundamental chamber G for the norm 1 roots.*

Proof. This is a straightforward computation. For example, the last ideal vertex $u := 4e_0 - 2(e_1 + e_2 + e_3) - (e_4 + e_5 + e_6 + e_7)$, which will be abbreviated $42^3 1^4$, satisfies $(u, \alpha_0 = e_0 - e_1 - e_2 - e_3) = -4 + 6 = 2$. Hence $s_0(u) = u - 2\alpha_0$ is conjugated under S_{13} to $u := 21^4$. Since $(u, \alpha_0) = -2 + 3 = 1$ we see that $s_0(u) = u - \alpha_0$ is conjugated under S_{13} to $u := 11$, which stands for $e_0 - e_1$. This is a cusp of D. Hence our original vertex u of P is separated from D by mirrors in norm 2 roots, and therefore u lies in G.

Using our compact notation the 18 cases can be conveniently summarized in the following table and the lemma follows from the observation that in each row the most right symbol represents a point in the chamber D for Γ^{13}.

$13A_1$	1			
	41^{13}			
$4A_1 \sqcup 3A_3$	21^3	1		
	$52^4 1^6$	421^9		
$A_1 \sqcup 4A_3$	31^8			
	431^4	321^2	21	
$2A_1 \sqcup A_2 \sqcup 3A_3$	321^3	21^2		
	$73^2 2^6 1$	$62^7 1^2$		
$A_1 \sqcup 3A_4$	$42^3 1^3$	21^3	1	
	$743^3 1^3$	431^4	321^2	21
$A_2 \sqcup A_3 \sqcup 2A_4$	$532^3 1^2$	321^3	21^2	
	$954^2 32^2 1$	$532^2 1^2$	31^3	
$3A_3 \sqcup A_4$	321^4	21^3	1	
	$63^2 2^3 1^3$	$42^2 1^5$	31^6	
$4D_4$	11			
	31^9			
$3A_5$	21^4	11		
	$42^3 1^4$	21^4	11	

Since we only used reflections s_i for $i = 0, 1, \cdots, 12$ in our reduction pattern we have in fact shown that

$$P \subset \cup_{w \in W(\mathrm{E}_{13})} wD \subset G$$

with E_{13} the subdiagram of the second Coxeter–Vinberg diagram in Theorem 1.1 with nodes numbered $0, 1, \cdots, 12$. $\qquad\square$

This finishes the proof of Theorem 1.3

5 Conclusions

The nodes of the I_{26} diagram are indexed by the set $\mathcal{I} = \mathcal{P} \sqcup \mathcal{L}$ of points and lines in $\mathbb{P}^2(3)$. Let us write $\omega = (-1 + \sqrt{-3})/2$ and $\theta = \sqrt{-3}$. The Allcock lattice L is a Lorentzian lattice over the ring of Eisenstein integers $\mathcal{E} = \mathbb{Z} + \mathbb{Z}\omega$ with generators ε_i for $i \in \mathcal{I}$ and Hermitian inner product

$$\langle \varepsilon_i, \varepsilon_i \rangle = 3, \langle \varepsilon_j, \varepsilon_k \rangle = 0, \langle \varepsilon_p, \varepsilon_l \rangle = \theta$$

for all $i, j, k \in \mathcal{I}$ with $j \neq k$ disconnected and all $p \in \mathcal{P}$ and $l \in \mathcal{L}$ connected in I_{26}. Note that $\langle \lambda, \mu \rangle \in \mathcal{E}\theta$ for all $\lambda, \mu \in L$. It is easy to see that L has rank 14 and signature $(13, 1)$ and so is Lorentzian.

A vector $\varepsilon \in L$ of norm $\langle \varepsilon, \varepsilon \rangle = 3$ is called a root. The order three complex reflection

$$t_\varepsilon(\lambda) = \lambda + (\omega - 1)\frac{\langle \lambda, \varepsilon \rangle}{\langle \varepsilon, \varepsilon \rangle}\varepsilon$$

is a unitary automorphism of L, and is called the triflection with root ε. It was shown by Allcock that the roots in L form a single orbit under $U(L)$, and Basak proved that $U(L)$ is generated by triflections [1],[6].

We extend scalars from $\mathcal{E}$ to $\mathbb{Z}[\zeta_{12}]$ and put

$$e_p = \varepsilon_p, e_l = -\sqrt{-1}\varepsilon_l$$

for $p \in \mathcal{P}$ and $l \in \mathcal{L}$. The Gram matrix of the $\{e_i\}$ becomes

$$\langle e_i, e_i \rangle = 3, \langle e_i, e_j \rangle = 0, \langle e_j, e_k \rangle = -\sqrt{3}$$

for all $i, j, k \in \mathcal{I}$ with $i \neq j$ disconnected and $j \neq k$ connected. The real vector space V spanned by the vectors $\{e_i; i \in \mathcal{I}\}$ is a Lorentzian vector space of dimension 14.

Lemma 5.1. *The intersection $L \cap V$ is an integral Lorentzian lattice with inner product values all contained in $3\mathbb{Z}$, and the integral lattice $L_r = L \cap V$ with the scalar product $(\cdot, \cdot) = \langle \cdot, \cdot \rangle / 3$ is isomorphic to the odd unimodular lattice $\mathbb{Z}^{13,1}$.*

Proof. Indeed, inner product values of $L \cap V$ lie in $\mathcal{E}\theta \cap \mathbb{R} = 3\mathbb{Z}$. Hence for a root $\varepsilon \in L$ either $e = \omega^j \varepsilon \in L_r$ for some j has norm 1 (for example for $\varepsilon = \varepsilon_p$ for some $p \in \mathcal{P}$) or $\omega^j \varepsilon \notin L_r$ for all j and $e = \omega^j \theta \varepsilon \in L_r$ for some j has norm 3 (for example for $\varepsilon = \varepsilon_l$ for some $l \in \mathcal{L}$). Since the discriminant of the Eisenstein lattice L is equal to 3^7 it follows that the discriminant of the integral lattice L_r should divide $3^{14}/3^{14} = 1$. Hence the lattice L_r is unimodular, odd, of signature $(13, 1)$ and so isomorphic to $\mathbb{Z}^{13,1}$ [22]. $\square$

The conclusion is that the intersection of the complex mirror arrangement in $\mathbb{C} \otimes_{\mathbb{R}} V$ for all norm three roots in L with the real form V consists of the mirror arrangement for all norm one roots in L_r together with the transform of this arrangement under the orthogonal involution of V coming from a projective duality on $\mathbb{P}^2(3)$. Hence the results of the previous section indeed prove Conjecture 1.7 made by one of us in [14]. Using the results of that paper we arrive at

Theorem 5.2. *Let* $\mathbb{B} = \{z \in \mathbb{C} \otimes_{\mathbb{R}} V; \langle z, z \rangle < 0\}/\mathbb{C}^{\times} \subset \mathbb{P}(\mathbb{C} \otimes_{\mathbb{R}} V)$ *be the complex hyperbolic ball and let* $\mathbb{B}^{\circ}$ *be the mirror arrangement complement. If* $\Gamma = \mathrm{PU}(L)$ *then the orbifold fundamental group* $\Pi_1^{\mathrm{orb}}(\mathbb{B}^{\circ}/\Gamma)$, *which according to a theorem of Allcock and Basak [3] is isomorphic to a factor group of the Artin group* $\mathrm{Art}(\mathrm{I}_{26})$ *with generators* T_i *for* $i \in \mathcal{I}$ *and braid relations*

$$T_i T_j = T_j T_i \quad , \quad T_k T_l T_k = T_l T_k T_l$$

for all $i, j, k, l \in \mathcal{I}$ *with* i, j *disconnected and* k, l *connected, has as factor group after imposing the Coxeter relations* $T_i^2 = 1$ *one of the following three groups, either* $M \wr 2 = (M \times M) \rtimes S_2$ *(the bimonster group) or* S_2 *or the trivial group.*

This provides a partial positive answer to the monstrous proposal of Daniel Allcock [2].

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